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EXCEPTIONALITY CONDITION AND LINEARIZATION OF HYPERBOLIC EQUATIONS

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1 Introduction

Nonlinear evolution phenomena in many physical contexts are described by nonlinear partial differential equations that, without loss of generality, we may assume in the form of first order systems.

Because of the difficulty of managing the nonlinearity, in order to characterize solutions or to point out qualitative results, it may be convenient to look for a suitable transformation of variables, when it exists, allowing to map the governing system to a "simpler form" that we know how to handle. A simpler form may be the autonomous form. (in which the independent variables do not appear explicitly in the coefficients) [1], [2] or, better, the linear form. In other words, one tries to characterize an invertible map, usually a point transformation, allowing us to obtain the expected result. A difficulty may arise when initial and/or boundary conditions are associated to the governing system, as these conditions may become very complicated under the linearizing transformation.

Usually the nonlinearity reflects on the transformation of variables as one gets a singularity in correspondence to the blow-up of the solution. This occurs, for instance, in the classical problem concerned with the hodograph transformation allowing to linearize 2×2 quasilinear, autonomous and homogeneous systems.

Hodograph-like transformations have been also used to linearize nonhomogeneous 2×2 systems [3]. Besides the fact that many *ad hoc* linearizing transformations are known which work with specific equations, another systematic approach to the linearization, usually employed for equations in conservative form, makes use of Bäcklund or reciprocal transformations [4], [5]. A combination of hodograph-like and Bäcklund transformations has been used to linearize some classes of 2×2 nonhomogeneous systems [6].

For what concerns the reduction to autonomous form we like to mention here the approach given by Dresner [7] connected to the search of similarity solutions of nonlinear partial differential equations [8], [9], [10], [11]. By looking for solutions invariant with respect to a one-parameter Lie group of point transformations representing a stretching group one gets equations with one independent variable less but which in general are nonautonomous even when the starting equations are autonomous. But, if these resulting equations are still invariant with respect to another one-parameter group of infinitesimal transformations (called by Dresner the "associated group"), then a further transformation of variables is available rendering them in autonomous form and with an independent variable less with respect to the original problem. A unified approach [12], [13] for the reduction either to autonomous form or to linear form makes use of Group Analysis for the governing system and requires the introduction of a convenient set of canonical variables, *i.e.* a set of privileged variables allowing to write every Lie group of infinitesimal transformations as a simple translation in only one variable. This transformation of variables, by choosing conveniently the new dependent and/or the new independent variables, allows us to reduce the system at hand to autonomous form or to linear form according to the structure of the Lie algebra associated with the Lie group of point transformations which leave the system at hand invariant. Specifically two theorems have been proved giving the algorithms of the desired reductions.

However, the application of this approach needs the knowledge of the group of symmetries of the governing system. Very often it is very complicated to accomplish this goal, also by using a computer algebra package [14], [15], especially in dealing with systems of equations involving constitutive functions which are not specified *a priori*. Consequently, it is very important to investigate whether there exist some additional criteria (besides the Lie group analysis).

In this paper particular emphasis will be given to linearizable systems connected with a special class of nonlinear systems known as completely exceptional or linearly degenerate [16], [17]. Linearly degenerate equations are, in some sense, similar to linear equations as the breakdown in the solutions due to the nonlinearity does not occur in a finite time.

Now, linear systems, especially when their coefficients are constant, can be reduced to a single higher order equation [18]. Consequently, if a nonlinear system can be linearized by an invertible map then it must be equivalent to a single nonlinear equation eventually linearly degenerate. In other words, if a nonlinear system is equivalent to a linear one with constant coefficients, as the latter can be transformed to a single higher order equation, then this equation must correspond through the same invertible map to a single nonlinear equation, eventually linearly degenerate, equivalent to the given nonlinear system. Such an approach has been shown to be valid in several cases [19] especially when the original system is described by a certain number of conservative equations. In fact, in this case it is possible to introduce suitable potentials transforming the system to a nonlinear equation that is completely exceptional. The further step is the linearization of completely exceptional equations of higher order. This problem may be solved for second and third order equations of the Monge-Ampère type [20], [21].

2 Linearization of 2×2 systems

If we suppose that an invertible map exists transforming nonlinear equations into linear ones then, because of the invertibility, the original nonlinear system must be such that the breakdown of the solution, due to the nonlinearity, will not occur as we want that solutions are transformed into solutions. But nonlinear systems having such a property are completely exceptional or linearly degenerate [16], [17].. Thus, in order to accomplish the linearization, we require first the condition of complete exceptionality to be satisfied. Moreover, as any linear system with constant coefficients can be transformed to a single higher order linear equation, always under the assumption of the invertibility of the transformation of variables, to that equation it must correspond a higher order nonlinear equation completely exceptional which is equivalent to the original system. We try to explain first on a specific example the reason which gives rise to the procedure we would like to develop.

Consider first a 2×2 hyperbolic homogeneous and autonomous system. In this case it is well known that Riemann invariants r, s exist such that the original system can be written under the form

$$r_t + \mu r_x = 0, \qquad s_t + \nu s_x = 0, \tag{2.1}$$

where $\mu(r,s)$ and $\nu(r,s)$ are the characteristic velocities associated to the original system. Now, if we consider the well known hodograph transformation

$$x = x(r,s), \qquad t = t(r,s), \tag{2.2}$$

the system (2.1) transforms into

$$x_s = \mu t_s, \qquad x_r = \nu t_r, \tag{2.3}$$

provided that the Jacobian of the hodograph transformation (2.2) is non-vanishing

$$J\left(\frac{\partial(x,t)}{\partial(r,s)}\right) = (\mu - \nu) t_r t_s \neq 0. \tag{2.4}$$

If t_r or t_s become zero along some characteristic curve then this will correspond to the breakdown of the solution due to the nonlinearity. The singularity of the Jacobian of the transformation will occur at some instant of time. The time corresponding to the first occurrence of such a singularity is said to be the critical time (t_c). Of course, by eliminating x in (2.3) we obtain a single linear second order equation which is equivalent to the original system while $t < t_c$. Consider now the case when the original 2×2 system is linearly degenerate; it is well known that both μ and ν are Riemann invariants and the system (2.3) in this case becomes [22]:

$$x_\mu = \mu t_\mu, \qquad x_\nu = \nu t_\nu \tag{2.5}$$

giving rise to the simple second order equation

$$t_{\mu\nu} = 0, \tag{2.6}$$

having as solution

$$t = T_1(\mu) + T_2(\nu), \tag{2.7}$$

the prime $'$ denoting differentiation with respect to the argument. Provided that $T_1'(\mu) \neq 0$ and $T_2'(\nu) \neq 0$ the hodograph transformation (2.2) is invertible and the original system is equivalent to a linear equation. When the system is genuinely nonlinear it can be shown [23] that either t_r or t_s become zero along some characteristic curve provided that some suitable conditions are verified. Now the nonlinear system corresponding to (2.5) is

$$\nu_t + \mu\nu_x = 0, \qquad \mu_t + \nu\mu_x = 0, \tag{2.8}$$

to which is associated the simple nonlinear second order equation

$$\frac{\partial}{\partial t}\left(\frac{\nu_t}{\nu_x}\right) + \nu \frac{\partial}{\partial x}\left(\frac{\nu_t}{\nu_x}\right) = 0, \tag{2.9}$$

that results to be linearly degenerate. In general, if the 2×2 system is not completely exceptional from the first of (2.1) by assuming $\mu_s \neq 0$ we may deduce $s = S\left(\frac{r_t}{r_x}, r\right)$ so that the second of (2.1) gives rise to

$$\frac{\partial}{\partial t} S\left(\frac{r_t}{r_x}, r\right) + \nu\left(S\left(\frac{r_t}{r_x}, r\right)\right) \frac{\partial}{\partial x} S\left(\frac{r_t}{r_x}, r\right) = 0, \tag{2.10}$$

which is a second order equation for r that is not in general completely exceptional. Assuming (2.10) to be hyperbolic the only condition rendering it completely exceptional is given by $\nu = \nu(r)$. The same conditions are valid if we require $\nu_r \neq 0$ in such a way from the second of (2.1) we get $r = R\left(\frac{s_t}{s_x}, s\right)$ that substituted into the first of (2.1) gives the second order equation for s

$$\frac{\partial}{\partial t} R\left(\frac{s_t}{s_x}, s\right) + \mu\left(R\left(\frac{s_t}{s_x}, s\right)\right) \frac{\partial}{\partial x} R\left(\frac{s_t}{s_x}, s\right) = 0, \tag{2.11}$$

which is completely exceptional only when $\mu = \mu(s)$. The corresponding first order systems which can be transformed to completely exceptional equations are in turn:

$$r_t + \mu(r,s) r_x = 0, \qquad s_t + \nu(r) s_x = 0, \tag{2.12}$$

and

$$r_t + \mu(s) r_x = 0, \qquad s_t + \nu(r,s) s_x = 0. \tag{2.13}$$

In the first system the characteristic velocity ν is exceptional, while in the second system μ has this property. Of course, the breakdown of the solution due to the nonlinearity will occur for the first system as t_r becomes zero along a characteristic propagating with velocity μ while for the second system this happens when t_s becomes zero along a characteristic propagating with velocity ν.

Now let us require the completely eceptional system (2.8) to be compatible with a supplementary conservation law of the type

$$\frac{\partial f(\mu, \nu)}{\partial t} + \frac{\partial g(\mu, \nu)}{\partial x} = 0. \tag{2.14}$$

The compatibility condition gives rise to an infinite number of possibilities with f and g having the expressions [24]

$$f(\mu,\nu)=\frac{M(\nu)-L(\mu)}{\mu-\nu},\qquad g(\mu,\nu)=\frac{\mu M(\nu)-\nu L(\mu)}{\mu-\nu}. \tag{2.15}$$

If we choose $L=\mu$, $M=\nu-1$ or $L=\mu$, $M=0$, we may write the system (2.8) in the following conservative form:

$$\frac{\partial}{\partial t}\frac{1}{\mu-\nu}+\frac{\partial}{\partial x}\frac{\mu}{\mu-\nu} = 0, \tag{2.16}$$

$$\frac{\partial}{\partial t}\frac{\mu}{\mu-\nu}+\frac{\partial}{\partial x}\frac{\mu\nu}{\mu-\nu} = 0. \tag{2.17}$$

As a consequence there exist two functions $\phi(x,t)$ and $\psi(x,t)$ such that

$$\frac{1}{\mu-\nu}=\phi_x,\qquad \frac{\mu}{\mu-\nu}=-\phi_t=\psi_x,\qquad \frac{\mu\nu}{\mu-\nu}=-\psi_t. \tag{2.18}$$

Again, we may introduce a further function $\theta(x,t)$ such that

$$\phi=\theta_x,\qquad \psi=-\theta_t, \tag{2.19}$$

whereupon we obtain:

$$\mu=-\frac{\theta_{xt}}{\theta_{xx}},\qquad \nu=-\frac{\theta_{xt}+1}{\theta_{xx}}, \tag{2.20}$$

with θ satisfying the Monge-Ampère equation [25]

$$\theta_{xx}\theta_{tt}-\theta_{xt}^2=\theta_{xt}, \tag{2.21}$$

which can be linearized as we shall see in the sequel.

We would like to stress here that the system (2.16)-(2.17), which is equivalent to the original system (2.8), is not only completely exceptional but also strictly exceptional, i.e., the only possible shock waves propagate with characteristic velocities. In fact, by taking the Rankine-Hugoniot relations across the shock line $\sigma(x,t)=0$ we get

$$\left[\frac{-\lambda+\mu}{\mu-\nu}\right]=0,\qquad \left[\frac{(-\lambda+\nu)\mu}{\mu-\nu}\right]=0, \tag{2.22}$$

where $\lambda = -\frac{\sigma_t}{\sigma_x}$ is the speed of propagation of shocks and $[\cdot] = (\cdot)_{\sigma=0^+} - (\cdot)_{\sigma=0^-}$ denotes the jump across the shock line. Taking into account the first relation, the first and the second relations may be written as follows:

$$\frac{-\lambda+\mu}{\mu-\nu}[\nu] = 0, \tag{2.23}$$

$$\frac{-\lambda+\nu}{\mu-\nu}[\mu] = 0. \tag{2.24}$$

So, if we require that $[\mu] \neq 0$ and $[\nu] \neq 0$ we have $\lambda = \nu$ and $\lambda = \mu$, i.e., the only admissible shock velocities are the characteristic velocities.

To a Monge-Ampère equation we are led in general by considering a 2×2 conservative system of the form

$$\begin{aligned} u_t + f_x &= 0, \\ f_t + g_x &= 0, \end{aligned} \tag{2.25}$$

where f and g are constitutive functions depending upon the unknown field variables u and v.

In order to show this property, let us consider the so called Martin procedure [26]. We first introduce two potential functions $\phi(x,t)$ and $\psi(x,t)$ such that

$$\phi_x = u, \qquad \phi_t = -f, \qquad \psi_x = f, \qquad g = -\psi_t. \tag{2.26}$$

Then the Martin potential θ defined by

$$d\theta = f dx + t dg = d\psi + d(tg) \tag{2.27}$$

is such that

$$\theta_x = f, \qquad \theta_g = t, \tag{2.28}$$

and in the new independent variables x and g there results:

$$\frac{\partial f}{\partial x} = \theta_{xx}, \qquad \frac{\partial f}{\partial g} = \theta_{xg}, \qquad \frac{\partial t}{\partial x} = \theta_{gx}, \qquad \frac{\partial t}{\partial g} = \theta_{gg}. \tag{2.29}$$

Consequently, we have

$$d\phi = (u - f\theta_{gx})\, dx - f\theta_{gg} dg, \tag{2.30}$$

and the compatibility condition provides

$$\theta_{xx}\theta_{gg} - \theta_{xg}^2 + u_g = 0 \tag{2.31}$$

that is the requested Monge-Ampère equation. Furthermore, it is easily shown that the condition in order $d\psi$ be a total differential is identically satisfied.

3 Linearization of completely exceptional equations

The linearization procedure we would like to develop consists in reducing first the governing system to a nonlinear higher order equation and then by requiring it to be completely exceptional. A further step allows us to linearize the equation so obtained.

Consider first a system of the form:

$$u_t + v_x = 0, \tag{3.1}$$

$$v_t + \frac{\partial}{\partial x} F(u, v, w) = 0, \tag{3.2}$$

$$u w_t + v w_x = 0. \tag{3.3}$$

¿From the equation (3.1) it follows that a function ϕ, playing the role of a potential, exists such that

$$u = \phi_x, \qquad v = -\phi_t. \tag{3.4}$$

Consequently, the equation (3.3) becomes

$$\phi_x w_t - \phi_t w_x = 0, \tag{3.5}$$

allowing us to write $w = w(\phi)$. From the equation (3.2) we finally obtain

$$\frac{\partial}{\partial t}\phi_t - \frac{\partial}{\partial x} F(\phi, \phi_x, \phi_t) = 0 \tag{3.6}$$

that is a conservative equation. A new potential $\psi(x,t)$ then exists such that

$$\phi_t = \psi_x, \qquad F = \psi_t, \tag{3.7}$$

but, due to the first of (3.7), there exists a further potential θ such that

$$\phi = \theta_x, \qquad \psi = \theta_t, \tag{3.8}$$

and we are able to write

$$\theta_{tt} = F(\theta_x, \theta_{xt}, \theta_{xx}). \tag{3.9}$$

We observe that discontinuities in the third order derivatives of θ will correspond to discontinuities in the second order derivatives of ϕ. Now, we require the equation (3.9) to be completely exceptional, i.e.,

$$\delta\lambda = 0 \quad \forall\lambda \tag{3.10}$$

where λ is the speed of propagation of weak discontinuities and $\delta = \frac{\partial}{\partial\varphi}\Big|_{\varphi=0^+} - \frac{\partial}{\partial\varphi}\Big|_{\varphi=0^-}$, $\varphi(x,t)$ being the curve across which the derivatives are discontinuous. The following functional form for F rendering the equation (3.9) completely exceptional has been obtained by Boillat [25]:

$$F = -\frac{-\alpha\theta_{xt}^2 + b\theta_{xt} + c\theta_{xx} + d}{\alpha\theta_{xx} + a}, \tag{3.11}$$

with the coefficients α, a, b, c and d representing arbitrary functions of θ_x. With the substitution $\phi = \theta_x$ we get the expression of $F(\phi_x, \phi_t, \phi)$ rendering the equation (3.6) completely exceptional. The equation (3.9) with F given by (3.11) produces the following Monge-Ampère equation:

$$\alpha(\theta_{xx}\theta_{tt} - \theta_{xt}^2) + a\theta_{tt} + b\theta_{xt} + c\theta_{xx} + d = 0. \tag{3.12}$$

where α, a, b, c, d represent arbitrary functions of θ_x.

The equation (3.6) becomes

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial x}\frac{-\alpha\phi_t^2 + b\phi_t + c\phi_x + d}{\alpha\phi_x + a} = 0 \tag{3.13}$$

which is linearly degenerate if C^2 weak discontinuities are considered.

The system (3.1)-(3.3) becomes

$$u_t + v_x = 0, \tag{3.14}$$

$$v_t + \frac{\partial}{\partial x}\frac{-\alpha v^2 - bv + au + d}{\alpha u + a} = 0, \tag{3.15}$$

$$uw_t + vw_x = 0, \tag{3.16}$$

where α, a, b, c, d represent arbitrary functions of w.

If $\alpha = 0$ we have

$$a\theta_{tt} + b\theta_{xt} + c\theta_{xx} + d = 0, \tag{3.17}$$

or

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial x}\frac{b\phi_t + c\phi_x + d}{a} = 0, \tag{3.18}$$

or

$$u_t + v_x = 0, \tag{3.19}$$

$$v_t + \frac{\partial}{\partial x}\frac{-bv + au + d}{a} = 0, \tag{3.20}$$

$$uw_t + vw_x = 0. \tag{3.21}$$

Equation (3.17) can be transformed into a 2×2 system

$$\phi_t - \psi_x = 0, \tag{3.22}$$

$$a\psi_t + b\psi_x + c\phi_x + d = 0, \tag{3.23}$$

where a, b, c, d represent arbitrary functions of ϕ. We have to look further for the linearization of (3.18) and (3.22)-(3.23). Most of the known linearizable equations or first order systems belong to (3.18) or (3.22)-(3.23). Before considering the linearization of (3.17) let us consider the general case when $\alpha \neq 0$.

In [20] there has been considered the reciprocal transformation

$$\tau = t - k_0\theta_t, \qquad \xi = x - k_1\theta_x, \qquad \theta^\star = -\theta + \frac{1}{2}(k_0\theta_t^2 + k_1\theta_x^2), \tag{3.24}$$

with k_0 and k_1 constant. If the relation

$$\alpha(k_0 c + k_1 a + 1) + k_0 k_1 d = 0 \tag{3.25}$$

is satisfied, then the Monge-Ampère equation can be mapped to the quasi-linear second order equation

$$\bar{a}\theta^\star_{\tau\tau} + \bar{b}\theta^\star_{\xi\tau} + \bar{c}\theta^\star_{\xi\xi} + \bar{d} = 0, \tag{3.26}$$

which is equivalent to the first order system

$$\begin{aligned} &\bar{a}(\phi)\psi_\tau + \bar{b}(\phi)\psi_\xi + \bar{c}(\phi)\phi_\xi - \bar{d} = 0, \\ &\phi_\tau - \psi_\xi = 0, \end{aligned} \tag{3.27}$$

with $\bar{a} = \alpha(a + k_0 d)$, $\bar{b} = \alpha(c + k_1 d)$, $\bar{c} = b$, $\bar{d} = -d$, that has the same form of the system corresponding to $\alpha = 0$.

The linearization of such a kind of 2×2 quasilinear systems has been considered by Seymour and Varley [3] by means of hodograph-like transformations.

Another way of linearizing the Monge-Ampère equation is the following. Let us introduce

$$\xi = \theta_x + kx, \qquad \tau = t, \qquad U = \theta_t, \tag{3.28}$$

with $k = \frac{a}{\alpha}$ constant. It results:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \xi}\theta_{xt}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}(\theta_{xx} + k). \tag{3.29}$$

It is recovered

$$U_\tau = \theta_{tt} - \frac{\theta_{xt}^2}{\theta_{xx}+k}, \qquad U_\xi = \frac{\theta_{xt}}{\theta_{xx}+k}, \tag{3.30}$$

and the Monge-Ampère equation reduces to

$$U_\tau + \bar{b}U_\xi + \frac{\bar{d}-\bar{c}k}{\theta_{xx}+k} + \bar{c} = 0, \tag{3.31}$$

where $\bar{b} = \frac{b}{\alpha}$, $\bar{c} = \frac{c}{\alpha}$, $\bar{d} = \frac{d}{\alpha}$. By taking the derivative with respect to τ we obtain the linear equation

$$U_{\tau\tau} + b_0 U_{\xi\tau} - h_0 U_{\xi\xi} + c_0 U_\xi = 0, \tag{3.32}$$

provided that

$$h_0 = \bar{d} - \bar{c}k, \qquad b_0 = \frac{b}{\alpha}, \qquad c_0 = \text{constant}. \tag{3.33}$$

4 A physical example

Let us consider the Euler equations for an isentropic fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{4.1}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + p(\rho, s))}{\partial x} = 0, \tag{4.2}$$

$$\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x} = 0, \tag{4.3}$$

where ρ is the fluid density, v the fluid velocity, s the entropy and $p(\rho, s)$ the pressure wich is a function of the density and the entropy. By introducing ϕ such that

$$\rho = \phi_x, \qquad \rho v = -\phi_t, \tag{4.4}$$

it results $s = s(\phi)$. Again by introducing ψ and θ such that

$$\begin{aligned} \rho v = \psi_x = -\phi_t, \qquad & \rho v^2 + p = -\psi_t, \\ \psi = -\theta_t, \qquad & \phi = \theta_x, \end{aligned} \tag{4.5}$$

we arrive to the nonlinear equation

$$\theta_{tt} = \frac{\theta_{xt}^2}{\theta_{xx}} + p(\theta_{xx}, s(\theta_x)). \tag{4.6}$$

This equation becomes of Monge-Ampère type for the class of fluids characterized by the constitutive law of Von Kármán [28]

$$p = -\frac{k^2(s)}{\rho} + b(s), \tag{4.7}$$

where $k^2(s) > 0$ and $b(s)$ are functions of the entropy. What we get is

$$\theta_{xx}\theta_{tt} - \theta_{xt}^2 + k^2(s(\theta_x)) - b(s(\theta_x))\theta_{xx} = 0. \tag{4.8}$$

In order to linearize the equation (4.8) we consider the Bäcklund transformation [19]

$$\xi = \theta_x, \qquad \tau = t, \qquad U = \theta_t. \tag{4.9}$$

Since

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \theta_{xt}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial x} = \theta_{xx}\frac{\partial}{\partial \xi}, \tag{4.10}$$

we obtain

$$\frac{\partial U}{\partial \tau} = \theta_{tt} - \frac{\theta_{xt}^2}{\theta_{xx}} = p(\theta_{xx}, s(\theta_x)), \qquad \frac{\partial U}{\partial \xi} = \frac{\theta_{xt}}{\theta_{xx}}. \tag{4.11}$$

Taking the derivative with respect to τ of the first of (4.11) and assuming (4.7) we arrive to the second order equation

$$U_{\tau\tau} = k^2(\xi)U_{\xi\xi}, \tag{4.12}$$

which is a linear equation with variable coefficients. The same procedure may be used to linearize the more general Monge-Ampère equation (3.12) provided that $a = 0$ and in this case we obtain

$$U_{\tau\tau} = A(\xi)U_{\xi\tau} + C(\xi)U_{\xi\xi}, \tag{4.13}$$

where $A = -b/\alpha$ and $C = -d/\alpha$.

By solving the equation (4.12) or the equation (4.13), we obtain the solution $U = U(\xi, \tau)$ from which we are led to the following first order nonlinear equation:

$$\theta_t = U(\theta_x, t) \tag{4.14}$$

that must be solved in order to obtain $\theta(x, t)$. Suppose now that the fluid equations (4.1)-(4.3) have to be solved along with the following initial conditions

$$\rho(x, 0) = \rho_0 = \text{constant}, \qquad v(x, 0) = 0, \qquad s(x, 0) = s_0(x). \tag{4.15}$$

As we have

$$\frac{\partial U}{\partial \tau} = p, \qquad \frac{\partial U}{\partial \xi} = \frac{\theta_{xt}}{\theta_{xx}} = v, \qquad \rho = \theta_{xx}, \qquad s = s(\xi), \tag{4.16}$$

the initial conditions to be associated with (4.12) are [27]:

$$U(\xi, 0) = 0, \qquad \frac{\partial U}{\partial \xi}(\xi, 0) = 0, \qquad \frac{\partial U}{\partial \tau}(\xi, 0) = p(s_0(\xi), \rho_0), \tag{4.17}$$

and the solution my be found by means of standard methods. For instance, if k^2 = constant in the constitutive law (4.7) we obtain the explicit solution

$$\begin{aligned} U(\xi, \tau) &= \frac{k^2}{\rho_0}\tau + \frac{1}{2k}[F(\xi - k\tau) - F(\xi + k\tau)], \\ F(\xi) &= \int b(\xi) d\xi. \end{aligned} \tag{4.18}$$

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